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Comments on Drinfeld Realization of Quantum Affine Superalgebra $U_q[gl(m|n)^{(1)}]$ and Its Hopf Algebra Structure

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Abstract

By generalizing the Reshetikhin and Semenov-Tian-Shansky construction to supersymmetric cases, we obtain Drinfeld current realization for quantum affine superalgebra $U_q[gl(m|n)^{(1)}]$. We find a simple coproduct for the quantum current generators and establish the Hopf algebra structure of this super current algebra.

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1 Introduction

This paper contains two results on the Drinfeld second realization (current realization) [1] for the untwisted quantum affine superalgebra $U_q[gl(m|n)^{(1)}]$.

The first result is the extension of Reshetikhin and Semenov-Tian-Shansky (RS) construction [2] to supersymmetric cases. Using this super RS construction and a super version of the Ding-Frenkel theorem [3], we obtain the defining relations for $U_q[gl(m|n)^{(1)}]$ in terms of super (or graded) current generators (generating functions). In [4], the authors made a similar effort. However, the relations they obtained are *not* supersymmetric and the algebra they defined is not a current realization of $U_q[gl(m|n)^{(1)}]$ but rather a current realization of a "nonstandard" quantum *bosonic* algebra associated with $U_q[gl(N = m+n)]$. This is because those authors failed to take care of the *grading* in the multiplication rule of tensor products which plays a fundamental role in any supersymmetric theories.

The second result is the coproduct, counit and antipode for the current realization of $U_q[gl(m|n)^{(1)}]$, thus establishing a Hopf algebra structure of this super current algebra.

2 Super RS Algebra and Ding-Frenkel Theorem

Let us start with introducing some useful notations. The graded Yang-Baxter equation (YBE) with spectral-parameter dependence takes the form

$$R_{12}(\frac{z}{w})R_{13}(z)R_{23}(w) = R_{23}(w)R_{13}(z)R_{12}(\frac{z}{w}), \tag{2.1}$$

where $R(z) \in End(V \otimes V)$ with V being graded vector space and obeys the weight conservation condition: $R(z)_{\alpha\beta}^{\alpha'\beta'} \neq 0$ only when $[\alpha'] + [\beta'] + [\alpha] + [\beta] = 0$ mod2. The multiplication rule for the tensor product is defined for homogeneous elements a, b, c, d of a quantum superalgebra by

$$(a \otimes b)(c \otimes d) = (-1)^{[b][c]}\,(ac \otimes bd) \tag{2.2}$$

where $[a] \in \mathbf{Z}_2$ denotes the grading of the element a.

Introduce the graded permutation operator P on the tensor product module $V \otimes V$ such that $P(v_\alpha \otimes v_\beta) = (-1)^{[\alpha][\beta]}(v_\beta \otimes v_\alpha)\,,\ \forall v_\alpha, v_\beta \in V$. In most cases R-matrix enjoys, among others, the following properties

$$(i)\ P_{12}R_{12}(z)P_{12} = R_{21}(z), \tag{2.3}$$

$$(ii)\ R_{12}(\frac{z}{w})R_{21}(\frac{w}{z}) = 1. \tag{2.4}$$

The graded YBE, when written in matrix form, carries extra signs, and depending on definition of matrix elements, these sign factors differ [5]. If we define the matrix elements of $R(z)$ by

$$R(z)(v^{\alpha'} \otimes v^{\beta'}) = R(z)_{\alpha\beta}^{\alpha'\beta'}(v^{\alpha} \otimes v^{\beta}), \tag{2.5}$$

then the matrix YBE reads [6]

$$\begin{aligned} &R(\frac{z}{w})_{\alpha\beta}^{\alpha'\beta'} R(z)_{\alpha'\gamma}^{\alpha''\gamma'} R(w)_{\beta'\gamma'}^{\beta''\gamma''} (-1)^{[\alpha][\beta]+[\gamma][\alpha']+[\gamma'][\beta']} \\ &\quad = R(w)_{\beta\gamma}^{\beta'\gamma'} R(z)_{\alpha\gamma'}^{\alpha'\gamma''} R(\frac{z}{w})_{\alpha'\beta'}^{\alpha''\beta''} (-1)^{[\beta][\gamma]+[\gamma'][\alpha]+[\beta'][\alpha']}. \end{aligned} \tag{2.6}$$

After the redefinition

$$\tilde{R}(z)_{\alpha\beta}^{\alpha'\beta'} = R(z)_{\alpha\beta}^{\alpha'\beta'} (-1)^{[\alpha][\beta]} \tag{2.7}$$

the extra signs in (2.6) disappear. However, this redefinition does not preserve semiclassical properties.

In matrix form the graded permutation operator $P = \sum_{\alpha,\beta} (-1)^{[\beta]} E_{\beta}^{\alpha} \otimes E_{\alpha}^{\beta}$ reads

$$P_{\alpha\beta}^{\alpha'\beta'} = \delta_{\alpha\beta'}\delta_{\alpha'\beta}(-1)^{[\alpha'][\beta']}. \tag{2.8}$$

The RS construction [2] can be generalized to supersymmetric cases. Formally, the super RS algebra is defined by similar relations as in non-supersymmetric cases [2, 3], but tensor products now carry gradings. We are thus led to

Definition 1 *: Super RS algebra is generated by invertible $L^{\pm}(z)$, satisfying*

$$\begin{aligned} R(\frac{z}{w})L_1^{\pm}(z)L_2^{\pm}(w) &= L_2^{\pm}(w)L_1^{\pm}(z)R(\frac{z}{w}), \\ R(\frac{z_+}{w_-})L_1^{+}(z)L_2^{-}(w) &= L_2^{-}(w)L_1^{+}(z)R(\frac{z_-}{w_+}), \end{aligned} \tag{2.9}$$

where $L_1^{\pm}(z) = L^{\pm}(z) \otimes 1$, $L_2^{\pm}(z) = 1 \otimes L^{\pm}(z)$ and $z_{\pm} = zq^{\pm\frac{c}{2}}$. For the first formula of (2.9), the expansion direction of $R(\frac{z}{w})$ can be chosen in $\frac{z}{w}$ or $\frac{w}{z}$, but for the second formula, the expansion direction must only be in $\frac{z}{w}$.

The super RS algebra is a Hopf algebra: its coproduct is defined by

$$\Delta(L^{\pm}(z) = L^{\pm}(zq^{\pm 1\otimes\frac{c}{2}}) \dot{\otimes} L^{\pm}(zq^{\mp\frac{c}{2}\otimes 1}), \tag{2.10}$$

and its antipode is

$$S(L^{\pm}(z)) = L^{\pm}(z)^{-1}. \tag{2.11}$$

In matrix form, (2.9) carries extra signs due to the graded multiplication rule of tensor products:

$$\begin{aligned}
&R(\frac{z}{w})_{\alpha\beta}^{\alpha"\beta"}\,L^{\pm}(z)_{\alpha"}^{\alpha'}\,L^{\pm}(w)_{\beta"}^{\beta'}\,(-1)^{[\alpha']([\beta']+[\beta"])}\\
&\qquad\qquad = L^{\pm}(w)_{\beta}^{\beta"}\,L^{\pm}(z)_{\alpha}^{\alpha"}\,R(\frac{z}{w})_{\alpha"\beta"}^{\alpha'\beta'}\,(-1)^{[\alpha]([\beta]+[\beta"])},\\
&R(\frac{z_+}{w_-})_{\alpha\beta}^{\alpha"\beta"}\,L^{+}(z)_{\alpha"}^{\alpha'}\,L^{-}(w)_{\beta"}^{\beta'}\,(-1)^{[\alpha']([\beta']+[\beta"])}\\
&\qquad\qquad = L^{-}(w)_{\beta}^{\beta"}\,L^{+}(z)_{\alpha}^{\alpha"}\,R(\frac{z_-}{w_+})_{\alpha"\beta"}^{\alpha'\beta'}\,(-1)^{[\alpha]([\beta]+[\beta"])}.
\end{aligned} \tag{2.12}$$

Introduce matrix θ:

$$\theta_{\alpha\beta}^{\alpha'\beta'} = (-1)^{[\alpha][\beta]}\delta_{\alpha\alpha'}\delta_{\beta\beta'}. \tag{2.13}$$

With the help of this matrix θ, one can cast (2.12) into the usual matrix equation,

$$\begin{aligned}
R(\frac{z}{w})L_1^{\pm}(z)\theta L_2^{\pm}(w)\theta &= \theta L_2^{\pm}(w)\theta L_1^{\pm}(z)R(\frac{z}{w}),\\
R(\frac{z_+}{w_-})L_1^{+}(z)\theta L_2^{-}(w)\theta &= \theta L_2^{-}(w)\theta L_1^{+}(z)R(\frac{z_-}{w_+}).
\end{aligned} \tag{2.14}$$

Now the multiplications in (2.14) are simply the usual matrix multiplications.

In this paper we take $R(\frac{z}{w}) \in End(V\otimes V)$ to be the R-matrix associated with $U_q[gl(m|n)]$, where V is a $(m+n)$-dimensional graded vector space. Let basis vectors $\{v^1,\ v^2,\cdots,\ v^m\}$ be even and $\{v^{m+1},\ v^{m+2},\cdots,\ v^{m+n}\}$ be odd. Then the R-matrix has the following matrix elements:

$$\begin{aligned}
R(\frac{z}{w})_{\alpha\beta}^{\alpha'\beta'} &= (-1)^{[\alpha][\beta]}\tilde{R}(\frac{z}{w})_{\alpha\beta}^{\alpha'\beta'},\\
\tilde{R}(\frac{z}{w}) &= \sum_{i=1}^{m}E_i^i\otimes E_i^i + \sum_{i=m+1}^{m+n}\frac{wq-zq^{-1}}{zq-wq^{-1}}E_i^i\otimes E_i^i + \frac{z-w}{zq-wq^{-1}}\sum_{i\neq j}(-1)^{[i][j]}E_i^i\otimes E_j^j\\
&\quad \sum_{i<j}\frac{z(q-q^{-1})}{zq-wq^{-1}}E_i^j\otimes E_j^i + \sum_{i>j}\frac{w(q-q^{-1})}{zq-wq^{-1}}E_i^j\otimes E_j^i.
\end{aligned} \tag{2.15}$$

It is easy to check that the R-matrix $R(z)$ satisfies (2.3) and (2.4). We will construct Drinfeld realization of $U_q[gl(m|n)^{(1)}]$. We first state a super version of the Ding-Frenkel theorem [3]:

Theorem 1 *: $L^{\pm}(z)$ has the following unique Gauss decomposition*

$$L^{\pm}(z) = \begin{pmatrix} 1 & \cdots & & 0\\ e_{2,1}^{\pm}(z) & \ddots & & \\ e_{3,1}^{\pm}(z) & & & \vdots\\ \vdots & & & \\ e_{m+n,1}^{\pm}(z) & \cdots & e_{m+n,m+n-1}^{\pm}(z) & 1 \end{pmatrix}\begin{pmatrix} k_1^{\pm}(z) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & k_{m+n}^{\pm}(z)\end{pmatrix}$$

$$\times \begin{pmatrix} 1 & f^{\pm}_{1,2}(z) & f^{\pm}_{1,3}(z) & \cdots & f^{\pm}_{1,m+n}(z) \\ \vdots & \ddots & \cdots & & \vdots \\ & & & & f^{\pm}_{m+n-1,m+n}(z) \\ 0 & & & & 1 \end{pmatrix}, \tag{2.16}$$

where $e^{\pm}_{i,j}(z),\ f^{\pm}_{j,i}(z)$ and $k^{\pm}_i(z)$ $(i>j)$ are elements in the super RS algebra and $k^{\pm}_i(z)$ are invertible. Let

$$\begin{aligned} X^-_i(z) &= f^+_{i,i+1}(z_+) - f^-_{i,i+1}(z_-), \\ X^+_i(z) &= e^+_{i+1,i}(z_-) - e^-_{i+1,i}(z_+), \end{aligned} \tag{2.17}$$

where $z_\pm = zq^{\pm\frac{c}{2}}$, then $q^{\pm\frac{c}{2}},\ X^{\pm}_i(z),\ k^{\pm}_j(z),\ i=1,2,\cdots,m+n-1,\ j=1,2,\cdots,m+n$ give the defining relations of quantum affine superalgebra $U_q[gl(m|n)^{(1)}]$.

The Gauss decomposition implies that the elements $e^{\pm}_{i,j}(z),\ f^{\pm}_{j,i}(z)$ $(i>j)$ and $k^{\pm}_i(z)$ are uniquely determined by $L^{\pm}(z)$. In the following we will denote $f^{\pm}_{i,i+1}(z),\ e^{\pm}_{i+1,i}(z)$ as $f^{\pm}_i(z),\ e^{\pm}_i(z)$, respectively.

The following matrix equations can be deduced from (2.14):

$$\begin{aligned} R_{21}(\frac{z}{w})\theta L^{\pm}_2(z)\theta L^{\pm}_1(w) &= L^{\pm}_1(w)\theta L^{\pm}_2(z)\theta R_{21}(\frac{z}{w}), & (2.18)\\ R_{21}(\frac{z_-}{w_+})\theta L^-_2(z)\theta L^+_1(w) &= L^+_1(w)\theta L^-_2(z)\theta R_{21}(\frac{z_+}{w_-}), & (2.19)\\ \theta L^{\pm}_2(z)^{-1}\theta L^{\pm}_1(w)^{-1}R_{21}(\frac{z}{w}) &= R_{21}(\frac{z}{w})L^{\pm}_1(w)^{-1}\theta L^{\pm}_2(z)^{-1}\theta, & (2.20)\\ \theta L^+_2(z)^{-1}\theta L^-_1(w)^{-1}R_{21}(\frac{z_+}{w_-}) &= R_{21}(\frac{z_-}{w_+})L^-_1(w)^{-1}\theta L^+_2(z)^{-1}\theta, & (2.21)\\ L^{\pm}_1(w)^{-1}R_{21}(\frac{z}{w})\theta L^{\pm}_2(z)\theta &= \theta L^{\pm}_2(z)\theta R_{21}(\frac{z}{w})L^{\pm}_1(w)^{-1}, & (2.22)\\ L^-_1(w)^{-1}R_{21}(\frac{z_+}{w_-})\theta L^+_2(z)\theta &= \theta L^+_2(z)\theta R_{21}(\frac{z_-}{w_+})L^-_1(w)^{-1}, & (2.23)\\ L^+_1(w)^{-1}R_{21}(\frac{z_-}{w_+})\theta L^-_2(z)\theta &= \theta L^-_2(z)\theta R_{21}(\frac{z_+}{w_-})L^+_1(w)^{-1}, & (2.24) \end{aligned}$$

where $R_{21}(\frac{z}{w}) = R(\frac{w}{z})^{-1}$. As in (2.14), the multiplications in (2.18 – 2.24) are usual matrix multiplications.

3 The $m=1,\ n=1$ Case: $U_q[gl(1|1)^{(1)}]$

For the simplest supersymmetric case $U_q[gl(1|1)^{(1)}]$, $L^{\pm}(z)$ take the forms

$$L^{\pm}(z) = \begin{pmatrix} k^{\pm}_1(z) & k^{\pm}_1(z)f^{\pm}_1(z) \\ e^{\pm}_1(z)k^{\pm}_1(z) & k^{\pm}_2(z) + e^{\pm}_1(z)k^{\pm}_1(z)f^{\pm}_1(z) \end{pmatrix}. \tag{3.1}$$

R-matrix and θ are given by,

$$R(\frac{z}{w}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{z-w}{zq-wq^{-1}} & \frac{z(q-q^{-1})}{zq-wq^{-1}} & 0 \\ 0 & \frac{w(q-q^{-1})}{zq-wq^{-1}} & \frac{z-w}{zq-wq^{-1}} & 0 \\ 0 & 0 & 0 & -\frac{wq-zq^{-1}}{zq-wq^{-1}} \end{pmatrix} \tag{3.2}$$

and $\theta = diag(1,1,1,-1)$, respectively.

Using (2.13), (2.15) and (2.14, 2.18 – 2.24), and by similar calculations as to the non-super case [3], we obtain a super RS algebra, which leads to, by means of theorem 1,

Definition 2 *: $U_q[gl(1|1)^{(1)}]$ is an associative algebra with unit 1 and the Drinfeld current generators: $X_1^{\pm}(z)$, $k_i^{\pm}(z)$ $(i=1,2)$, a central element c and a nonzero complex parameter q. $k_i^{\pm}(z)$ are invertible. The gradings of the generators are: $[X_1^{\pm}(z)] = 1$ and $[k_i^{\pm}(z)] = 0 = [c]$. The relations read,*

$$\begin{aligned} k_i^{\pm}(z)k_j^{\pm}(w) &= k_j^{\pm}(w)k_i^{\pm}(z), \quad i,\ j=1,2, \\ k_1^{+}(z)k_1^{-}(w) &= k_1^{-}(w)k_1^{+}(z), \\ \frac{w_-q - z_+q^{-1}}{z_+q - w_-q^{-1}}k_2^{+}(z)k_2^{-}(w) &= \frac{w_+q - z_-q^{-1}}{z_-q - w_+q^{-1}}k_2^{-}(w)k_2^{+}(z), \\ \frac{z_\pm - w_\mp}{z_\pm q - w_\mp q^{-1}}k_2^{\mp}(w)^{-1}k_1^{\pm}(z) &= \frac{z_\mp - w_\pm}{z_\mp q - w_\pm q^{-1}}k_1^{\pm}(z)k_2^{\mp}(w)^{-1}, \\ k_i^{\pm}(z)^{-1}X_1^{-}(w)k_i^{\pm}(z) &= \frac{z_\mp q - wq^{-1}}{z_\mp - w}X_1^{-}(w), \\ k_i^{\pm}(z)X_1^{+}(w)k_i^{\pm}(z)^{-1} &= \frac{z_\pm q - wq^{-1}}{z_\pm - w}X_1^{+}(w), \\ \{X_1^{\pm}(z), X_1^{\pm}(w)\} &= 0, \\ \{X_1^{+}(z), X_1^{-}(w)\} &= (q-q^{-1})\Big(\delta(\frac{w}{z}q^{c})k_2^{+}(w_+)k_1^{+}(w_+)^{-1} \\ &\qquad -\delta(\frac{w}{z}q^{-c})k_2^{-}(z_+)k_1^{-}(z_+)^{-1}\Big), \end{aligned} \tag{3.3}$$

where $\{X,Y\} \equiv XY + YX$ denotes an anti-commutator and

$$\delta(z) = \sum_{k\in\mathbf{Z}} z^k \tag{3.4}$$

is a formal series.

Theorem 2 *: The algebra $U_q[gl(1|1)^{(1)}]$ defined by (3.3) has a Hopf algebra structure, which is given by the following formulae.*

Coproduct Δ

$$
\begin{aligned}
&\Delta(q^c) = q^c \otimes q^c,\\
&\Delta(k_i^+(z)) = k_i^+(zq^{\frac{c_2}{2}}) \otimes k_i^+(zq^{-\frac{c_1}{2}}),\\
&\Delta(k_i^-(z)) = k_i^-(zq^{-\frac{c_2}{2}}) \otimes k_i^-(zq^{\frac{c_1}{2}}),\\
&\Delta(X_1^+(z)) = X_1^+(z) \otimes 1 + \psi_1(zq^{\frac{c_1}{2}}) \otimes X_1^+(zq^{c_1}),\\
&\Delta(X_1^-(z)) = 1 \otimes X_1^-(z) + X_1^-(zq^{c_2}) \otimes \phi_1(zq^{\frac{c_2}{2}}),
\end{aligned}
\tag{3.5}
$$

where $c_1 = c \otimes 1$, $c_2 = 1 \otimes c$, $\psi_1(z) = k_2^-(z)k_1^-(z)^{-1}$ and $\phi_1(z) = k_2^+(z)k_1^+(z)^{-1}$.

Counit ϵ

$$
\epsilon(q^c) = 1, \qquad \epsilon(k_i^\pm(z)) = 1, \qquad \epsilon(X_1^\pm(z)) = 0.
\tag{3.6}
$$

Antipode S

$$
\begin{aligned}
&S(q^c) = q^{-c}, \qquad S(k_i^\pm(z)) = k_i^\pm(z)^{-1},\ i = 1,2,\\
&S(X_1^+(z)) = -\psi_1(zq^{-\frac{c}{2}})^{-1}X_1^+(zq^{-c}),\\
&S(X_1^-(z)) = -X_1^-(zq^{-c})\phi_1(zq^{-\frac{c}{2}})^{-1}.
\end{aligned}
\tag{3.7}
$$

Proof. The proof is rather elementary. We nevertheless present the details since the proof for the general case in next section is similar. Care has to be taken of the gradings in tensor product multiplications and also in extending the antipode to the whole algebra.

$$
\begin{aligned}
\Delta\left(\frac{w_-q - z_+q^{-1}}{z_+q - w_-q^{-1}}k_2^+(z)k_2^-(w)\right) &= \frac{wq^{\frac{-c_1-c_2}{2}+1} - zq^{\frac{c_1+c_2}{2}-1}}{zq^{\frac{c_1+c_2}{2}+1} - wq^{\frac{-c_1-c_2}{2}-1}}\\
&\times \left(k_2^+(zq^{\frac{c_2}{2}}) \otimes k_2^+(zq^{-\frac{c_1}{2}})\right)\left(k_2^-(wq^{-\frac{c_2}{2}}) \otimes k_2^-(wq^{\frac{c_1}{2}})\right)\\
&= \frac{wq^{\frac{c_1+c_2}{2}+1} - zq^{\frac{-c_1-c_2}{2}-1}}{zq^{\frac{-c_1-c_2}{2}+1} - wq^{\frac{c_1+c_2}{2}-1}}\\
&\times \left(k_2^-(wq^{-\frac{c_2}{2}}) \otimes k_2^-(wq^{\frac{c_1}{2}})\right)\left(k_2^+(zq^{\frac{c_2}{2}}) \otimes k_2^+(zq^{-\frac{c_1}{2}})\right)\\
&= \Delta\left(\frac{w_+q - z_-q^{-1}}{z_-q - w_+q^{-1}}k_2^-(w)k_2^+(z)\right),
\end{aligned}
\tag{3.8}
$$

where the relation between $k_2^+(z)$ and $k_2^-(w)$ has been used.

$$
\begin{aligned}
\Delta\left(\frac{z_+ - w_-}{z_+q - w_-q^{-1}}k_1^+(z)k_2^-(w)\right) &= \frac{zq^{\frac{c_1+c_2}{2}} - wq^{\frac{-c_1-c_2}{2}}}{zq^{\frac{c_1+c_2}{2}+1} - wq^{\frac{-c_1-c_2}{2}-1}}\\
&\times \left(k_1^+(zq^{\frac{c_2}{2}}) \otimes k_1^+(zq^{-\frac{c_1}{2}})\right)\left(k_2^-(wq^{-\frac{c_2}{2}}) \otimes k_2^-(wq^{\frac{c_1}{2}})\right)\\
&= \frac{zq^{\frac{-c_1-c_2}{2}} - wq^{\frac{c_1+c_2}{2}}}{zq^{\frac{-c_1-c_2}{2}+1} - wq^{\frac{c_1+c_2}{2}-1}}\\
&\times \left(k_2^-(wq^{-\frac{c_2}{2}}) \otimes k_2^-(wq^{\frac{c_1}{2}})\right)\left(k_1^+(zq^{\frac{c_2}{2}}) \otimes k_1^+(zq^{-\frac{c_1}{2}})\right)\\
&= \Delta\left(\frac{z_- - w_+}{z_-q - w_+q^{-1}}k_2^-(w)k_1^+(z)\right).
\end{aligned}
\tag{3.9}
$$

The relation between $\Delta(k_1^-(z))$ and $\Delta(k_2^+(w))$ can be proved along a simliar line.

The fifth and sixth equations in (3.3) are equivalent to the relations,

$$\begin{aligned}\phi_1(z)X_1^{\pm}(w)\phi_1(z)^{-1} = X_1^{\pm}(w),\\ \psi_1(z)X_1^{\pm}(w)\psi_1(z)^{-1} = X_1^{\pm}(w)\end{aligned} \tag{3.10}$$

which are easily seen to be preserved by the coproduct, thanks to the commutativity of $\phi_1(z)$ and $\psi_1(w)$.

$$\begin{aligned}\Delta\left(\{X_1^+(z), X_1^+(w)\}\right) &= \Delta(X_1^+(z))\Delta(X_1^+(w)) + \Delta(X_1^+(w))\Delta(X_1^+(z))\\ &= X_1^+(z)X_1^+(w)\otimes 1 + X_1^+(z)\psi_1(wq^{\frac{c_1}{2}})\otimes X_1^+(wq^{c_1})\\ &\quad -\psi_1(zq^{\frac{c_1}{2}})X_1^+(w)\otimes X_1^+(zq^{c_1})\\ &\quad +X_1^+(w)X_1^+(z)\otimes 1 + X_1^+(w)\psi_1(zq^{\frac{c_1}{2}})\otimes X_1^+(zq^{c_1})\\ &\quad -\psi_1(wq^{\frac{c_1}{2}})X_1^+(z)\otimes X_1^+(wq^{c_1})\\ &\quad +\psi_1(zq^{\frac{c_1}{2}})\psi_1(wq^{\frac{c_1}{2}})\otimes X_1^+(zq^{c_1})X_1^+(wq^{c_1})\\ &\quad +\psi_1(wq^{\frac{c_1}{2}})\psi_1(zq^{\frac{c_1}{2}})\otimes X_1^+(wq^{c_1})X_1^+(zq^{c_1})\\ &= \{X_1^+(z), X_1^+(w)\}\otimes 1 + \psi_1(zq^{\frac{c_1}{2}})\psi_1(wq^{\frac{c_1}{2}})\\ &\quad \otimes\{X_1^+(zq^{c_1}), X_1^+(wq^{c_1})\} = 0.\end{aligned} \tag{3.11}$$

$\Delta(\{X_1^-(z), X_1^-(w)\}) = 0$ can be proved similarly.

$$\begin{aligned}\Delta\left(\{X_1^+(z), X_1^-(w)\}\right) &= \Delta(X_1^+(z))\Delta(X_1^-(w)) + \Delta(X_1^-(w))\Delta(X_1^+(z))\\ &= X_1^+(z)\otimes X_1^-(w) + X_1^+(z)X_1^-(wq^{c_2})\otimes\phi_1(wq^{\frac{c_2}{2}})\\ &\quad +\psi_1(zq^{\frac{c_1}{2}})\otimes X_1^+(zq^{c_1})X_1^-(w)\\ &\quad -X_1^+(z)\otimes X_1^-(w) + \psi_1(zq^{\frac{c_1}{2}})\otimes X_1^-(w)X_1^+(zq^{c_1})\\ &\quad +X_(^- wq^{c_2})X_1^+(z)\otimes\phi_1(wq^{\frac{c_2}{2}})\\ &\quad -\psi_1(zq^{\frac{c_1}{2}})X_1^-(wq^{c_2})\otimes X_1^+(zq^{c_1})\phi_1(wq^{\frac{c_2}{2}})\\ &\quad +X_1^-(wq^{c_2})\psi_1(zq^{\frac{c_1}{2}})X_1^+(zq^{c_1})\\ &= [X_1^+(z), X_1^-(wq^{c_2})]\otimes\phi_1(wq^{c_2}) + \psi_1(zq^{\frac{c_1}{2}})\otimes[X_1^+(zq^{c_1}), X_1^-(w)]\\ &= (q-q^{-1})\left(\delta(\frac{w}{z}q^{c_1+c_2})\phi_1(wq^{\frac{c_2}{2}+\frac{c_1+c_2}{2}})\phi_1(wq^{-\frac{c_1}{2}+\frac{c_1+c_2}{2}})\right.\\ &\quad \left.-\delta(\frac{w}{z}q^{-c_1-c_2})\psi_1(zq^{\frac{-c_2}{2}+\frac{c_1+c_2}{2}})\psi_1(wq^{\frac{c_1}{2}+\frac{c_1+c_2}{2}})\right)\\ &= (q-q^{-1})\Delta\left(\delta(\frac{w}{z}q^{c})\phi_1(w_+) - \delta(\frac{w}{z}q^{-c})\psi_1(z_+)\right).\end{aligned} \tag{3.12}$$

We have therefore proved that the comultiplication is an algebra homomorphism.

$$S\left(\{X_1^+(z), X_1^-(w)\}\right) = -S(X_1^-(w))S(X_1^+(z)) - S(X_1^+(z))S(X_1^-(w))$$

$$
\begin{aligned}
&= -\psi_1(zq^{-\frac{c}{2}})^{-1}\phi_1(wq^{-\frac{c}{2}})^{-1}[X_1^+(zq^{-c}), X_1^-(wq^{-c})] \\
&= -\psi_1(zq^{-\frac{c}{2}})^{-1}\phi_1(wq^{-\frac{c}{2}})^{-1}(q-q^{-1}) \\
&\quad \times \left(\delta(\frac{w}{z}q^{c})\phi_1(w^{-\frac{c}{2}}) - \delta(\frac{w}{z}q^{-c})\psi_1(zq^{-\frac{c}{2}})\right) \\
&= (q-q^{-1})\left(\delta(\frac{w}{z}q^{-c})\phi_1(w^{-\frac{c}{2}})^{-1} - \delta(\frac{w}{z}q^{c})\psi_1(zq^{-\frac{c}{2}})^{-1}\right) \\
&= (q-q^{-1})S\left(\delta(\frac{w}{z}q^{c})\phi_1(w^{\frac{c}{2}}) - \delta(\frac{w}{z}q^{-c})\psi_1(zq^{\frac{c}{2}})\right). \qquad (3.13)
\end{aligned}
$$

We can prove in the same manner that other relations are also preserved by the antipode.

Let $M: U_q[gl(1|1)^{(1)}] \otimes U_q[gl(1|1)^{(1)}] \rightarrow U_q[gl(1|1)^{(1)}]$ be a multiplication. Then we can easily check

$$
\begin{aligned}
&M(1\otimes\epsilon)\Delta = id = M(\epsilon\otimes 1)\Delta, \\
&M(1\otimes S)\Delta = \epsilon = M(S\otimes 1)\Delta. \qquad (3.14)
\end{aligned}
$$

Thus we have shown that the coproduct, the counit and the antipode give a Hopf algebra structure.

4 General Case: $U_q[gl(m|n)^{(1)}]$

The generalization to the general case $U_q[gl(m|n)^{(1)}]$ is more or less straightforward by using (2.15), (2.14, 2.18 – 2.24) and theorem 1. Similar to the non-supersymmetric cases [3, 4], this is achieved by induction on m and n. Tedious but direct computations give rise to

Definition 3 *: $U_q[gl(m|n)^{(1)}]$ is an associative algebra with unit 1 and Drinfeld current generators: $X_i^{\pm}(z)$, $k_j^{\pm}(z)$, $i = 1, 2, \cdots, m+n-1$, $j = 1, 2, \cdots, m+n$, a central element c and a nonzero complex parameter q. $k_i^{\pm}(z)$ are invertible. The grading of the generators are: $[X_m^{\pm}(z)] = 1$ and zero otherwise. The defining relations are given by*

$$
\begin{aligned}
k_i^{\pm}(z)k_j^{\pm}(w) &= k_j^{\pm}(w)k_i^{\pm}(z), \quad i \neq j \\
k_i^{+}(z)k_i^{-}(w) &= k_i^{-}(w)k_i^{+}(z), \quad i \leq m, \\
\frac{w_-q - z_+q^{-1}}{z_+q - w_-q^{-1}}k_i^{+}(z)k_i^{-}(w) &= \frac{w_+q - z_-q^{-1}}{z_-q - w_+q^{-1}}k_i^{-}(w)k_i^{+}(z), \quad m < i \leq m+n, \\
\frac{z_\pm - w_\mp}{z_\pm q - w_\mp q^{-1}}k_i^{\mp}(w)^{-1}k_j^{\pm}(z) &= \frac{z_\mp - w_\pm}{z_\mp q - w_\pm q^{-1}}k_j^{\pm}(z)k_i^{\mp}(w)^{-1}, \quad i > j, \\
k_j^{\pm}(z)^{-1}X_i^{-}(w)k_j^{\pm}(z) &= X_i^{-}(w), \quad j-i \leq -1, \\
k_j^{\pm}(z)^{-1}X_i^{+}(w)k_j^{\pm}(z) &= X_i^{+}(w), \quad j-i \leq -1, \quad \text{or} \\
k_j^{\pm}(z)^{-1}X_i^{-}(w)k_j^{\pm}(z) &= X_i^{-}(w), \quad j-i \geq 2,
\end{aligned}
$$

$$
\begin{aligned}
k_j^{\pm}(z)^{-1}X_i^+(w)k_j^{\pm}(z) &= X_i^+(w), \quad j-i\geq 2,\\
k_i^{\pm}(z)^{-1}X_i^-(w)k_i^{\pm}(z) &= \frac{z_{\mp}q-wq^{-1}}{z_{\mp}-w}X_i^-(w), \quad i<m,\\
k_i^{\pm}(z)^{-1}X_i^-(w)k_i^{\pm}(z) &= \frac{z_{\mp}q^{-1}-wq}{z_{\mp}-w}X_i^-(w), \quad m<i\leq m+n-1,\\
k_{i+1}^{\pm}(z)^{-1}X_i^-(w)k_{i+1}^{\pm}(z) &= \frac{z_{\mp}q^{-1}-wq}{z_{\mp}-w}X_i^-(w), \quad i<m,\\
k_{i+1}^{\pm}(z)^{-1}X_i^-(w)k_{i+1}^{\pm}(z) &= \frac{z_{\mp}q-wq^{-1}}{z_{\mp}-w}X_i^-(w), \quad m<i\leq m+n-1,\\
k_i^{\pm}(z)X_i^+(w)k_i^{\pm}(z)^{-1} &= \frac{z_{\pm}q-wq^{-1}}{z_{\pm}-w}X_i^+(w), \quad i<m,\\
k_i^{\pm}(z)X_i^+(w)k_i^{\pm}(z)^{-1} &= \frac{z_{\pm}q^{-1}-wq}{z_{\pm}-w}X_i^+(w), \quad m<i\leq m+n-1,\\
k_{i+1}^{\pm}(z)X_i^+(w)k_{i+1}^{\pm}(z)^{-1} &= \frac{z_{\pm}q^{-1}-wq}{z_{\pm}-w}X_i^+(w), \quad i<m,\\
k_{i+1}^{\pm}(z)X_i^+(w)k_{i+1}^{\pm}(z)^{-1} &= \frac{z_{\pm}q-wq^{-1}}{z_{\pm}-w}X_i^+(w), \quad m<i\leq m+n-1,\\
k_i^{\pm}(z)^{-1}X_m^-(w)k_i^{\pm}(z) &= \frac{z_{\mp}q-wq^{-1}}{z_{\mp}-w}X_m^-(w), \quad i=m,\ m+1,\\
k_i^{\pm}(z)X_m^+(w)k_i^{\pm}(z)^{-1} &= \frac{z_{\pm}q-wq^{-1}}{z_{\pm}-w}X_m^+(w), \quad i=m,\ m+1,\\
(zq^{\mp1}-wq^{\pm1})X_i^{\mp}(z)X_i^{\mp}(w) &= (zq^{\pm1}-wq^{\mp1})X_i^{\mp}(w)X_i^{\mp}(z), \quad i<m,\\
(wq^{\mp1}-zq^{\pm1})X_i^{\mp}(z)X_i^{\mp}(w) &= (wq^{\pm1}-zq^{\mp1})X_i^{\mp}(w)X_i^{\mp}(z), \quad m<i\leq m+n-1,\\
\{X_m^{\pm}(z),X_m^{\pm}(w)\} &= 0,\\
(z-w)X_i^+(z)X_{i+1}^+(w) &= (zq-wq^{-1})X_{i+1}^+(w)X_i^+(z), \quad i<m,\\
(w-z)X_i^+(z)X_{i+1}^+(w) &= (wq-zq^{-1})X_{i+1}^+(w)X_i^+(z), \quad m\leq i\leq m+n-1,\\
(zq-wq^{-1})X_i^-(z)X_{i+1}^-(w) &= (z-w)X_{i+1}^-(w)X_i^-(z), \quad i<m,\\
(wq-zq^{-1})X_i^-(z)X_{i+1}^-(w) &= (w-z)X_{i+1}^-(w)X_i^-(z), \quad m\leq i\leq m+n-1,\\
[X_i^+(z),X_j^-(w)] &= -(q-q^{-1})\delta_{ij}\left(\delta(\frac{w}{z}q^c)k_{i+1}^+(w_+)k_i^+(w_+)^{-1}\right.\\
&\qquad \left.-\delta(\frac{w}{z}q^{-c})k_{i+1}^-(z_+)k_i^-(z_+)^{-1}\right), \quad i,j\neq m,\\
\{X_m^+(z),X_m^-(w)\} &= (q-q^{-1})\left(\delta(\frac{w}{z}q^c)k_{m+1}^+(w_+)k_m^+(w_+)^{-1}\right.\\
&\qquad \left.-\delta(\frac{w}{z}q^{-c})k_{m+1}^-(z_+)k_m^-(z_+)^{-1}\right),
\end{aligned}
\tag{4.1}
$$

where $[X,Y]\equiv XY-YX$ stands for a commutator and $\{X,Y\}\equiv XY+YX$ for an anti-commutator, together with the following Serre and extra Serre [7, 8] relations:

$$
\begin{aligned}
&\{X_i^{\pm}(z_1)X_i^{\pm}(z_2)X_{i+1}^{\pm}(w)-(q+q^{-1})X_i^{\pm}(z_1)X_{i+1}^{\pm}(w)X_i^{\pm}(z_2)\\
&\qquad +X_{i+1}^{\pm}(w)X_i^{\pm}(z_1)X_i^{\pm}(z_2)\}+\{z_1\leftrightarrow z_2\}=0, \quad i\neq m,
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
&\{X^{\pm}_{i+1}(z_1)X^{\pm}_{i+1}(z_2)X^{\pm}_{i}(w)-(q+q^{-1})X^{\pm}_{i+1}(z_1)X^{\pm}_{i}(w)X^{\pm}_{i+1}(z_2)\\
&\qquad +X^{\pm}_{i}(w)X^{\pm}_{i+1}(z_1)X^{\pm}_{i+1}(z_2)\}+\{z_1\leftrightarrow z_2\}=0,\quad i\neq m-1, \qquad (4.3)\\
&\{(z_1q^{\mp 1}-z_2q^{\pm 1})[X^{\pm}_{m}(z_1)X^{\pm}_{m}(z_2)X^{\pm}_{m-1}(w)-(q+q^{-1})X^{\pm}_{m}(z_1)X^{\pm}_{m-1}(w)X^{\pm}_{m}(z_2)\\
&\qquad +X^{\pm}_{m-1}(w)X^{\pm}_{m}(z_1)X^{\pm}_{m}(z_2)]\}+\{z_1\leftrightarrow z_2\}=0, \qquad (4.4)\\
&\{(z_2q^{\mp 1}-z_1q^{\pm 1})[X^{\pm}_{m}(z_1)X^{\pm}_{m}(z_2)X^{\pm}_{m+1}(w)-(q+q^{-1})X^{\pm}_{m}(z_1)X^{\pm}_{m+1}(w)X^{\pm}_{m}(z_2)\\
&\qquad +X^{\pm}_{m+1}(w)X^{\pm}_{m}(z_1)X^{\pm}_{m}(z_2)]\}+\{z_1\leftrightarrow z_2\}=0, \qquad (4.5)\\
&\{(z_1q^{\mp 1}-z_2q^{\pm 1})[X^{\pm}_{m}(z_1)X^{\pm}_{m}(z_2)X^{\pm}_{m-1}(w_1)X^{\pm}_{m+1}(w_2)\\
&\qquad -(q+q^{-1})X^{\pm}_{m}(z_1)X^{\pm}_{m-1}(w_1)X^{\pm}_{m}(z_2)X^{\pm}_{m+1}(w_2)]\\
&\qquad +(z_1+z_2)(q^{\mp 1}-q^{\pm 1})X^{\pm}_{m-1}(w_1)X^{\pm}_{m}(z_1)X^{\pm}_{m}(z_2)X^{\pm}_{m+1}(w_2)\\
&\qquad +(z_2q^{\mp 1}-z_1q^{\pm 1})[-(q+q^{-1})X^{\pm}_{m-1}(w_1)X^{\pm}_{m}(z_1)X^{\pm}_{m+1}(w_2)X^{\pm}_{m}(z_2)\\
&\qquad +X^{\pm}_{m-1}(w_1)X^{\pm}_{m+1}(w_2)X^{\pm}_{m}(z_1)X^{\pm}_{m}(z_2)]\}+\{z_1\leftrightarrow z_2\}=0. \qquad (4.6)
\end{aligned}
$$

Theorem 3 *: The algebra $U_q[gl(m|n)^{(1)}]$ given by definition 3 has a Hopf algebra structure, which is given by the following formulae.*

Coproduct Δ

$$
\begin{aligned}
&\Delta(q^c)=q^c\otimes q^c,\\
&\Delta(k^+_j(z))=k^+_j(zq^{\frac{c_2}{2}})\otimes k^+_j(zq^{-\frac{c_1}{2}}),\\
&\Delta(k^-_j(z))=k^-_j(zq^{-\frac{c_2}{2}})\otimes k^-_j(zq^{\frac{c_1}{2}}),\quad j=1,2,\cdots,m+n\\
&\Delta(X^+_i(z))=X^+_i(z)\otimes 1+\psi_i(zq^{\frac{c_1}{2}})\otimes X^+_i(zq^{c_1}),\\
&\Delta(X^-_i(z))=1\otimes X^-_i(z)+X^-_i(zq^{c_2})\otimes\phi_i(zq^{\frac{c_2}{2}}),\quad i=1,2,\cdots,m+n-1, \qquad (4.7)
\end{aligned}
$$

where $c_1=c\otimes 1,\ c_2=1\otimes c,\ \psi_i(z)=k^-_{i+1}(z)k^-_i(z)^{-1}$ *and* $\phi_i(z)=k^+_{i+1}(z)k^+_i(z)^{-1}$.

Counit ϵ

$$
\epsilon(q^c)=1,\qquad \epsilon(k^{\pm}_j(z))=1,\qquad \epsilon(X^{\pm}_i(z))=0. \qquad (4.8)
$$

Antipode S

$$
\begin{aligned}
&S(q^c)=q^{-c},\qquad S(k^{\pm}_j(z))=k^{\pm}_j(z)^{-1},\\
&S(X^+_i(z))=-\psi_i(zq^{-\frac{c}{2}})^{-1}X^+_i(zq^{-c}),\\
&S(X^-_i(z))=-X^-_i(zq^{-c})\phi_i(zq^{-\frac{c}{2}})^{-1}. \qquad (4.9)
\end{aligned}
$$

Proof: Similar to the case in the previous section for $U_q[gl(1|1)^{(1)}]$, this theorem is proved by direct calculations.

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References

[1] V.G. Drinfeld, Sov. Math. Dokl. **36** (1988) 212.

[2] N.Yu. Reshetikhin, M.A. Semenov-Tian-Shansky, Lett. Math. Phys. **19** (1990) 133.

[3] J. Ding, I.B. Frenkel, Commun. Math. Phys. **155** (1993) 277.

[4] H. Fan, B.-Y. Hou, K.-J. Shi, J. Math. Phys. **38** (1997) 411.

[5] Y.-Z. Zhang, *On the graded Yang-Baxter and reflection equations*, Commun. Theor. Phys. (1996).

[6] A.J. Bracken, G.W. Delius, M.D. Gould, Y.-Z. Zhang, J. Phys. **A:** Math. Gen. **27** (1994) 6551.

[7] M. Scheunert, Lett. Math. Phys. **24** (1992) 173.

[8] H. Yamane, *On defining relations of affine Lie superalgebras and their quantized universal enveloping superalgebras*, preprint q-alg/9603015.